

# SYMMETRIES OF SECOND ORDER DIFFERENTIAL EQUATIONS ON LIE ALGEBROIDS

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**ABSTRACT.** In this paper we investigate the relations between semispray, nonlinear connection, dynamical covariant derivative and Jacobi endomorphism on Lie algebroids. Using these geometric structures, we study the symmetries of second order differential equations in the general framework of Lie algebroids.

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## 1. Introduction

The geometry of second order differential equations (SODE) on the tangent bundle  $TM$  of a differentiable manifold  $M$  is closely related to the geometry of nonlinear connections [8, 15]. The system of SODE can be represented using the notion of semispray, which together with the nonlinear connection induce two important concepts: the dynamical covariant derivative and Jacobi endomorphism [4, 5, 9, 16, 25, 36]. The notion of symmetry in fields theory using various geometric framework are intensely studied (see for instance [1, 3, 5, 12, 18, 20, 35]). The notion of Lie algebroid is a natural generalization of the tangent bundle. In the last decades the Lie algebroids [23, 24] are the objects of intensive studies with applications to mechanical systems or optimal control ([2, 7, 10, 19, 22, 26, 27, 28, 30, 31, 32, 33, 34, 38]) and are the natural framework in which one can develop the theory of differential equations, where the notion of symmetry plays a very important role.

In this paper we study some properties of semispray and generalize the notion of symmetry for second order differential equations on Lie algebroids and characterize its properties using the dynamical covariant derivative and Jacobi endomorphism. The paper is organized as follows. In the section two the preliminary geometric structures on Lie algebroids are introduced and some relations between them are given. We present the Jacobi endomorphism on Lie algebroids and find the relation with the curvature tensor of Ehresmann nonlinear connection. In the section three we study the dynamical covariant derivative on Lie algebroids. Using a semispray and an arbitrary nonlinear connection, we introduce the dynamical covariant derivative on Lie algebroids as a tensor derivation and prove that the compatibility condition with the tangent structure fix the canonical nonlinear connection. In the case of the canonical nonlinear connection induced by a semispray, more properties of dynamical covariant derivative are added. In the case of homogeneous second order differential equations (spray) the relation between the dynamical covariant derivative and Berwald connection is given. In the last section we study the dynamical symmetry, Lie symmetry, Newtonoid section and Cartan symmetry on Lie algebroids and find the relations between them. Moreover, we characterize their

properties in terms of dynamical covariant derivative and Jacobi endomorphism, which generalize some results from [5]. We have to mention that the Noether's theorem for Lagrangian systems on Lie algebroids can be found in [6]. Also, using the  $k$ -symplectic formalism on Lie algebroids developed in [21] one can study the symmetries in this new framework, which generalize the results from [3].

## 2. Lie algebroids

Let  $M$  be a real,  $C^\infty$ -differentiable,  $n$ -dimensional manifold and  $(TM, \pi_M, M)$  its tangent bundle. A Lie algebroid over a manifold  $M$  is a triple  $(E, [\cdot, \cdot]_E, \sigma)$ , where  $(E, \pi, M)$  is a vector bundle of rank  $m$  over  $M$ , which satisfies the following conditions:

- a)  $C^\infty(M)$ -module of sections  $\Gamma(E)$  is equipped with a Lie algebra structure  $[\cdot, \cdot]_E$ .
- b)  $\sigma : E \rightarrow TM$  is a bundle map (called the anchor) which induces a Lie algebra homomorphism (also denoted  $\sigma$ ) from the Lie algebra of sections  $(\Gamma(E), [\cdot, \cdot]_E)$  to the Lie algebra of vector fields  $(\chi(M), [\cdot, \cdot])$  satisfying the Leibniz rule

$$(1) \quad [s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2, \quad \forall s_1, s_2 \in \Gamma(E), \quad f \in C^\infty(M).$$

From the above definition it results:

- 1°  $[\cdot, \cdot]_E$  is a  $\mathbb{R}$ -bilinear operation,
- 2°  $[\cdot, \cdot]_E$  is skew-symmetric, i.e.  $[s_1, s_2]_E = -[s_2, s_1]_E$ ,  $\forall s_1, s_2 \in \Gamma(E)$ ,
- 3°  $[\cdot, \cdot]_E$  verifies the Jacobi identity

$$[s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0,$$

and  $\sigma$  being a Lie algebra homomorphism, means that  $\sigma[s_1, s_2]_E = [\sigma(s_1), \sigma(s_2)]$ .

The existence of a Lie bracket on the space of sections of a Lie algebroid leads to a calculus on its sections analogous to the usual Cartan calculus on differential forms.

If  $f$  is a function on  $M$ , then  $df(x) \in E_x^*$  is given by  $\langle df(x), a \rangle = \sigma(a)f$ , for  $\forall a \in E_x$ . For  $\omega \in \bigwedge^k(E^*)$  the *exterior derivative*  $d^E\omega \in \bigwedge^{k+1}(E^*)$  is given by the formula

$$\begin{aligned} d^E\omega(s_1, \dots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \sigma(s_i) \omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1}) + \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([s_i, s_j]_E, s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1}). \end{aligned}$$

where  $s_i \in \Gamma(E)$ ,  $i = \overline{1, k+1}$ , and the hat over an argument means the absence of the argument. It results that

$$(d^E)^2 = 0, \quad d^E(\omega_1 \wedge \omega_2) = d^E\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d^E\omega_2,$$

The cohomology associated with  $d^E$  is called the *Lie algebroid cohomology* of  $E$ . Also, for  $\xi \in \Gamma(E)$  one can define the *Lie derivative* with respect to  $\xi$  by  $\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$ , where  $i_\xi$  is the contraction with  $\xi$ .

We recall that if  $L$  and  $K$  are  $(1, 1)$ -type tensor field, Frölicher-Nijenhuis bracket  $[L, K]$  is the vector valued 2-form [11]

$$\begin{aligned} [L, K]_E(X, Y) &= [LX, KY]_E + [KX, LY]_E + (LK + KL)[X, Y]_E - \\ &\quad - L[X, KY]_E - K[X, LY]_E - L[KX, Y]_E - K[LX, Y]_E. \end{aligned}$$

and the Nijenhuis tensor of  $L$  is given by

$$\mathbf{N}_L(X, Y) = \frac{1}{2}[L, L]_E = [LX, LY]_E + L^2[X, Y]_E - L[X, LY]_E - L[LX, Y]_E.$$

For a vector field in  $\mathcal{X}(E)$  and a  $(1, 1)$ -type tensor field  $L$  on  $E$  the Frölicher-Nijenhuis bracket  $[X, L]_E = \mathcal{L}_X L$  is the  $(1, 1)$ -type tensor field on  $E$  given by

$$\mathcal{L}_X L = \mathcal{L}_X \circ L - L \circ \mathcal{L}_X,$$

where  $\mathcal{L}_X$  is the usual Lie derivative.

If we take the local coordinates  $(x^i)$  on an open  $U \subset M$ , a local basis  $\{s_\alpha\}$  of the sections of the bundle  $\pi^{-1}(U) \rightarrow U$  generates local coordinates  $(x^i, y^\alpha)$  on  $E$ . The local functions  $\sigma_\alpha^i(x)$ ,  $L_{\alpha\beta}^\gamma(x)$  on  $M$  given by

$$(2) \quad \sigma(s_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L_{\alpha\beta}^\gamma s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

are called the *structure functions of the Lie algebroid*, and satisfy the *structure equations* on Lie algebroids

$$\sum_{(\alpha, \beta, \gamma)} \left( \sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\delta}{\partial x^i} + L_{\alpha\eta}^\delta L_{\beta\gamma}^\eta \right) = 0, \quad \sigma_\alpha^j \frac{\partial \sigma_\beta^i}{\partial x^j} - \sigma_\beta^j \frac{\partial \sigma_\alpha^i}{\partial x^j} = \sigma_\gamma^i L_{\alpha\beta}^\gamma.$$

Locally, if  $f \in C^\infty(M)$  then  $d^E f = \frac{\partial f}{\partial x^i} \sigma_\alpha^i s^\alpha$ , where  $\{s^\alpha\}$  is the dual basis of  $\{s_\alpha\}$  and if  $\theta \in \Gamma(E^*)$ ,  $\theta = \theta_\alpha s^\alpha$  then

$$d^E \theta = \left( \sigma_\alpha^i \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2} \theta_\gamma L_{\alpha\beta}^\gamma \right) s^\alpha \wedge s^\beta.$$

Particularly, we get  $d^E x^i = \sigma_\alpha^i s^\alpha$  and  $d^E s^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha s^\beta \wedge s^\gamma$ .

**2.1. The prolongation of a Lie algebroid over the vector bundle projection.** Let  $(E, \pi, M)$  be a vector bundle. For the projection  $\pi : E \rightarrow M$  we can construct the prolongation of  $E$  (see [17, 19, 26]). The associated vector bundle is  $(\mathcal{T}E, \pi_2, E)$  where

$$\mathcal{T}E = \bigcup_{w \in E} \mathcal{T}_w E, \quad \mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w \pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection  $\pi_2(u_x, v_w) = \pi_E(v_w) = w$ , where  $\pi_E : TE \rightarrow E$  is the tangent projection. We also have the canonical projection  $\pi_1 : \mathcal{T}E \rightarrow E$  given by  $\pi_1(u, v) = u$ . The projection onto the second factor  $\sigma^1 : \mathcal{T}E \rightarrow TE$ ,  $\sigma^1(u, v) = v$  will be the anchor of a new Lie algebroid over the manifold  $E$ . An element of  $\mathcal{T}E$  is said to be vertical if it is in the kernel of the projection  $\pi_1$ . We will denote  $(V\mathcal{T}E, \pi_2|_{V\mathcal{T}E}, E)$  the vertical bundle of  $(\mathcal{T}E, \pi_2, E)$  and  $\sigma^1|_{V\mathcal{T}E} : V\mathcal{T}E \rightarrow VTE$  is an isomorphism. If  $f \in C^\infty(M)$  we will denote by  $f^c$  and  $f^v$  the *complete and vertical lift* to  $E$  of  $f$  defined by

$$f^c(u) = \sigma(u)(f), \quad f^v(u) = f(\pi(u)), \quad u \in E.$$

For  $s \in \Gamma(E)$  we can consider the *vertical lift* of  $s$  given by  $s^v(u) = s(\pi(u))_u^v$ , for  $u \in E$ , where  $\frac{v}{u} : E_{\pi(u)} \rightarrow T_u(E_{\pi(u)})$  is the canonical isomorphism. There exists a unique vector field  $s^c$  on  $E$ , the *complete lift* of  $s$  satisfying the following conditions:

- i)  $s^c$  is  $\pi$ -projectable on  $\sigma(s)$ ,
- ii)  $s^c(\hat{\alpha}) = \widehat{\mathcal{L}_s \alpha}$ ,

for all  $\alpha \in \Gamma(E^*)$ , where  $\hat{\alpha}(u) = \alpha(\pi(u))(u)$ ,  $u \in E$  (see [13, 14]).

Considering the prolongation  $\mathcal{T}E$  of  $E$  ([26]), we may introduce the *vertical lift*  $s^v$  and the *complete lift*  $s^c$  of a section  $s \in \Gamma(E)$  as the sections of  $\mathcal{T}E \rightarrow E$  given by

$$s^v(u) = (0, s^v(u)), \quad s^c(u) = (s(\pi(u)), s^c(u)), \quad u \in E.$$

Other two canonical objects on  $\mathcal{T}E$  are the *Euler section*  $\mathbb{C}$  and the *tangent structure* (vertical endomorphism)  $J$ . The Euler section  $\mathbb{C}$  is the section of  $\mathcal{T}E \rightarrow E$  defined by  $\mathbb{C}(u) = (0, u_u^v)$ ,  $\forall u \in E$ . The vertical endomorphism is the section of  $(\mathcal{T}E) \oplus (\mathcal{T}E)^* \rightarrow E$  characterized by  $J(s^v) = 0$ ,  $J(s^c) = s^v$ ,  $s \in \Gamma(E)$  which satisfies

$$J^2 = 0, \quad \text{Im} J = \ker J = V\mathcal{T}E, \quad [\mathbb{C}, J]_{\mathcal{T}E} = -J.$$

A section  $\mathcal{S}$  of  $\mathcal{T}E \rightarrow E$  is called *semispray* (*second order differential equation -SODE*) on  $E$  if  $J(\mathcal{S}) = \mathbb{C}$ . The local basis of  $\Gamma(\mathcal{T}E)$  is given by  $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ , where [26]

$$(3) \quad \mathcal{X}_\alpha(u) = \left( s_\alpha(\pi(u)), \sigma_\alpha^i \frac{\partial}{\partial x^i} \Big|_u \right), \quad \mathcal{V}_\alpha(u) = \left( 0, \frac{\partial}{\partial y^\alpha} \Big|_u \right),$$

and  $(\partial/\partial x^i, \partial/\partial y^\alpha)$  is the local basis on  $TE$ . The structure functions of  $\mathcal{T}E$  are given by the following formulas

$$(4) \quad \sigma^1(\mathcal{X}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha},$$

$$(5) \quad [\mathcal{X}_\alpha, \mathcal{X}_\beta]_{\mathcal{T}E} = L_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0.$$

The vertical lift of a section  $\rho = \rho^\alpha s_\alpha$  is  $\rho^v = \rho^\alpha \mathcal{V}_\alpha$ . The coordinate expression of Euler section is  $\mathbb{C} = y^\alpha \mathcal{V}_\alpha$  and the local expression of  $J$  is given by  $J = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$ , where  $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$  denotes the corresponding dual basis of  $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ . The Nijenhuis tensor of the vertical endomorphism vanishes and it results that  $J$  is integrable. The expression of the complete lift of a section  $\rho = \rho^\alpha s_\alpha$  is

$$(6) \quad \rho^c = \rho^\alpha \mathcal{X}_\alpha + (\sigma_\varepsilon^i \frac{\partial \rho^\alpha}{\partial x^i} - L_{\beta\varepsilon}^\alpha \rho^\beta) y^\varepsilon \mathcal{V}_\alpha.$$

In particular  $s_\alpha^v = \mathcal{V}_\alpha$ ,  $s_\alpha^c = \mathcal{X}_\alpha - L_{\alpha\varepsilon}^\beta y^\varepsilon \mathcal{V}_\beta$ . The local expression of the differential of a function  $L$  on  $\mathcal{T}E$  is  $d^E L = \sigma_\alpha^i \frac{\partial L}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha} \mathcal{V}^\alpha$  and we have  $d^E x^i = \sigma_\alpha^i \mathcal{X}^\alpha$ ,  $d^E y^\alpha = \mathcal{V}^\alpha$ . The differential of sections of  $(\mathcal{T}E)^*$  is determined by

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E \mathcal{V}^\alpha = 0.$$

In local coordinates a semispray has the expression

$$(7) \quad \mathcal{S}(x, y) = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha(x, y) \mathcal{V}_\alpha.$$

and the following equality holds

$$(8) \quad J[\mathcal{S}, JX]_{\mathcal{T}E} = -JX, \quad X \in \Gamma(E).$$

The integral curves of  $\sigma^1(\mathcal{S})$  satisfy the differential equations

$$\frac{dx^i}{dt} = \sigma_\alpha^i(x) y^\alpha, \quad \frac{dy^\alpha}{dt} = \mathcal{S}^\alpha(x, y).$$

If we have the relation  $[\mathbb{C}, \mathcal{S}]_{\mathcal{T}E} = \mathcal{S}$  then  $\mathcal{S}$  is called *spray* and the functions  $\mathcal{S}^\alpha$  are homogeneous functions of degree 2 in  $y^\alpha$ .

For a regular Lagrangian  $L$  on  $E$ , that is the matrix

$$g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$$

has constant rank  $m$ , the symplectic structure induced by  $L$  is given by [26]

$$\omega_L = g_{\alpha\beta} \mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2} \left( \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

Considering the energy function  $E_L = \mathbb{C}(L) - L$ , with local expression

$$E_L = y^\alpha \frac{\partial L}{\partial y^\alpha} - L,$$

then the symplectic equation

$$i_S \omega_L = -d^E E_L,$$

determines the components of the semispray [26]

$$S^\varepsilon = g^{\varepsilon\beta} \left( \sigma_\beta^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L_{\beta\alpha}^\gamma y^\alpha \frac{\partial L}{\partial y^\gamma} \right),$$

where  $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$ , which depends only on the regular Lagrangian and the structure function of the Lie algebroid.

## 2.2. Nonlinear connections on Lie algebroids.

**Definition 1.** A nonlinear connection on  $\mathcal{TE}$  is an almost product structure  $\mathcal{N}$  on  $\pi_2 : \mathcal{TE} \rightarrow E$  (i.e. a bundle morphism  $\mathcal{N} : \mathcal{TE} \rightarrow \mathcal{TE}$ , such that  $\mathcal{N}^2 = Id$ ) smooth on  $\mathcal{TE} \setminus \{0\}$  such that  $V\mathcal{TE} = \ker(Id + \mathcal{N})$ .

If  $\mathcal{N}$  is a connection on  $\mathcal{TE}$  then  $H\mathcal{TE} = \ker(Id - \mathcal{N})$  is the horizontal subbundle associated to  $\mathcal{N}$  and  $\mathcal{TE} = V\mathcal{TE} \oplus H\mathcal{TE}$ . Each  $\rho \in \Gamma(\mathcal{TE})$  can be written as  $\rho = \rho^h + \rho^v$ , where  $\rho^h, \rho^v$  are sections in the horizontal and respective vertical subbundles. If  $\rho^h = 0$ , then  $\rho$  is called *vertical* and if  $\rho^v = 0$ , then  $\rho$  is called *horizontal*. A connection  $\mathcal{N}$  on  $\mathcal{TE}$  induces two projectors  $h, v : \mathcal{TE} \rightarrow \mathcal{TE}$  such that  $h(\rho) = \rho^h$  and  $v(\rho) = \rho^v$  for every  $\rho \in \Gamma(\mathcal{TE})$ . We have

$$h = \frac{1}{2}(Id + \mathcal{N}), \quad v = \frac{1}{2}(Id - \mathcal{N}), \quad \ker h = Im v = V\mathcal{TE}, \quad Im h = \ker v = H\mathcal{TE}.$$

$$h^2 = h, \quad v^2 = v, \quad hv = vh = 0, \quad h + v = Id, \quad h - v = \mathcal{N}.$$

$$Jh = J, \quad hJ = 0, \quad Jv = 0, \quad vJ = J.$$

Locally, a connection can be expressed as  $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}_\alpha^\beta \mathcal{V}_\beta$ ,  $\mathcal{N}(\mathcal{V}_\beta) = -\mathcal{V}_\beta$ , where  $\mathcal{N}_\alpha^\beta = \mathcal{N}_\alpha^\beta(x, y)$  are the local coefficients of  $\mathcal{N}$ . The sections

$$(9) \quad \delta_\alpha = h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta \mathcal{V}_\beta,$$

generate a basis of  $H\mathcal{TE}$ . The frame  $\{\delta_\alpha, \mathcal{V}_\alpha\}$  is a local basis of  $\mathcal{TE}$  called Berwald basis. The dual adapted basis is  $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$  where  $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta$ . The Lie brackets of the adapted basis  $\{\delta_\alpha, \mathcal{V}_\alpha\}$  are [31]

$$(10) \quad [\delta_\alpha, \delta_\beta]_{\mathcal{TE}} = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta]_{\mathcal{TE}} = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} \mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{TE}} = 0,$$

$$(11) \quad \mathcal{R}_{\alpha\beta}^\gamma = \delta_\beta(\mathcal{N}_\alpha^\gamma) - \delta_\alpha(\mathcal{N}_\beta^\gamma) + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma.$$

**Definition 2.** *The curvature of the nonlinear connection  $\mathcal{N}$  on  $\mathcal{TE}$  is  $\Omega = -\mathbf{N}_h$  where  $h$  is the horizontal projector and  $\mathbf{N}_h$  is the Nijenhuis tensor of  $h$ .*

In local coordinates we have

$$\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta}^{\gamma}\mathcal{X}^{\alpha}\wedge\mathcal{X}^{\beta}\otimes\mathcal{V}_{\gamma},$$

where  $\mathcal{R}_{\alpha\beta}^{\gamma}$  are given by (11) and represent the local coordinate functions of the curvature tensor. The curvature of the nonlinear connection is an obstruction to the integrability of  $H\mathcal{TE}$ , understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of  $\mathcal{TE}$ . The horizontal distribution  $H\mathcal{TE}$  is integrable if and only if the curvature  $\Omega$  of the nonlinear connection vanishes. Also, from the Jacobi identity we obtain

$$[h, \Omega]_{\mathcal{TE}} = 0.$$

Let us consider a semispray  $\mathcal{S}$  and an arbitrary nonlinear connection  $\mathcal{N}$  with induced  $(h, v)$  projectors. Then we set (see also [33])

**Definition 3.** *The vertically valued  $(1, 1)$ -type tensor field on Lie algebroid  $\mathcal{TE}$  given by*

$$(12) \quad \Phi = -v \circ \mathcal{L}_{\mathcal{S}}v,$$

*will be called the Jacobi endomorphism.*

We obtain

$$\Phi = -v \circ \mathcal{L}_{\mathcal{S}}v = v \circ \mathcal{L}_{\mathcal{S}}h = v \circ (\mathcal{L}_{\mathcal{S}} \circ h - h \circ \mathcal{L}_{\mathcal{S}}) = v \circ \mathcal{L}_{\mathcal{S}} \circ h.$$

and in local coordinates the action of Lie derivative on the Berwald basis is given by

$$(13) \quad \mathcal{L}_{\mathcal{S}}\delta_{\beta} = (\mathcal{N}_{\beta}^{\alpha} - L_{\beta\varepsilon}^{\alpha}y^{\varepsilon})\delta_{\alpha} + \mathcal{R}_{\beta}^{\gamma}\mathcal{V}_{\gamma}, \quad \mathcal{L}_{\mathcal{S}}\mathcal{V}_{\beta} = -\delta_{\beta} - \left(\mathcal{N}_{\beta}^{\alpha} + \frac{\partial\mathcal{S}^{\alpha}}{\partial y^{\beta}}\right)\mathcal{V}_{\alpha}.$$

The Jacobi endomorphism has the local form

$$(14) \quad \Phi = \mathcal{R}_{\beta}^{\alpha}\mathcal{V}_{\alpha}\otimes\mathcal{X}^{\beta}, \quad \mathcal{R}_{\beta}^{\gamma} = -\sigma_{\beta}^i\frac{\partial\mathcal{S}^{\gamma}}{\partial x^i} - \mathcal{S}(\mathcal{N}_{\beta}^{\gamma}) + \mathcal{N}_{\beta}^{\alpha}\mathcal{N}_{\alpha}^{\gamma} + \mathcal{N}_{\beta}^{\alpha}\frac{\partial\mathcal{S}^{\gamma}}{\partial y^{\alpha}} + \mathcal{N}_{\varepsilon}^{\gamma}L_{\alpha\beta}^{\varepsilon}y^{\alpha}.$$

**Proposition 1.** *The following formula holds*

$$\Phi = i_{\mathcal{S}}\Omega + v \circ \mathcal{L}_{\mathcal{S}}h.$$

**Proof.** Indeed,  $\Phi(\rho) = v \circ \mathcal{L}_{\mathcal{S}}h\rho = v \circ \mathcal{L}_{h\mathcal{S}}h\rho + v \circ \mathcal{L}_{v\mathcal{S}}h\rho$  and  $\Omega(\mathcal{S}, \rho) = v[h\mathcal{S}, h\rho]_{\mathcal{TE}} = v \circ \mathcal{L}_{h\mathcal{S}}h\rho$ , which yields  $\Phi(\rho) = \Omega(\mathcal{S}, \rho) + v \circ \mathcal{L}_{v\mathcal{S}}h\rho$ .  $\square$

**Proposition 2.** *If  $\mathcal{S}$  is a spray, then the Jacobi endomorphism is the contraction with  $\mathcal{S}$  of curvature of the nonlinear connection*

$$\Phi = i_{\mathcal{S}}\Omega.$$

**Proof.** If  $\mathcal{S}$  is a spray, then the coefficients  $\mathcal{S}^{\alpha}$  are 2-homogeneous with respect to the variables  $y^{\beta}$  and it results

$$(15) \quad \mathcal{S} = h\mathcal{S} = y^{\alpha}\delta_{\alpha}, \quad v\mathcal{S} = 0, \quad \mathcal{N}_{\beta}^{\alpha} = \frac{\partial\mathcal{N}_{\varepsilon}^{\alpha}}{\partial y^{\beta}}y^{\varepsilon} + L_{\beta\varepsilon}^{\alpha}y^{\varepsilon},$$

which yields  $\Phi = i_S \Omega$  and locally we get  $\mathcal{R}_\beta^\alpha = \mathcal{R}_{\varepsilon\beta}^\alpha y^\varepsilon$ , which represents the local relation between the Jacobi endomorphism and the curvature of the nonlinear connection.

### 3. Dynamical covariant derivative on Lie algebroids

In following we will introduce the notion of dynamical covariant derivative on Lie algebroids as a tensor derivation and study its properties.

**Definition 4.** [33] *A map  $\nabla : \mathfrak{T}(\mathcal{TE} \setminus \{0\}) \rightarrow \mathfrak{T}(\mathcal{TE} \setminus \{0\})$  is said to be a tensor derivation on  $\mathcal{TE} \setminus \{0\}$  if the following conditions are satisfied:*

- i)  $\nabla$  is  $\mathbb{R}$ -linear*
- ii)  $\nabla$  is type preserving, i.e.  $\nabla(\mathfrak{T}_s^r(\mathcal{TE} \setminus \{0\})) \subset \mathfrak{T}_s^r(\mathcal{TE} \setminus \{0\})$ , for each  $(r, s) \in \mathbb{N} \times \mathbb{N}$ .*
- iii)  $\nabla$  obeys the Leibnitz rule  $\nabla(P \otimes S) = \nabla P \otimes S + P \otimes \nabla S$ , for any tensors  $P, S$  on  $\mathcal{TE} \setminus \{0\}$ .*
- iv)  $\nabla$  commutes with any contractions, where  $\mathfrak{T}_\bullet(\mathcal{TE} \setminus \{0\})$  is the space of tensors on  $\mathcal{TE} \setminus \{0\}$ .*

For a semispray  $\mathcal{S}$  and an arbitrary nonlinear connection  $\mathcal{N}$  we consider the  $\mathbb{R}$ -linear map  $\nabla : \Gamma(\mathcal{TE} \setminus \{0\}) \rightarrow \Gamma(\mathcal{TE} \setminus \{0\})$  given by

$$(16) \quad \nabla = h \circ \mathcal{L}_\mathcal{S} \circ h + v \circ \mathcal{L}_\mathcal{S} \circ v,$$

which will be called the dynamical covariant derivative induced by the semispray  $\mathcal{S}$  and the nonlinear connection  $\mathcal{N}$ . By setting  $\nabla f = \mathcal{S}(f)$ , for  $f \in C^\infty(\mathcal{TE} \setminus \{0\})$  using the Leibnitz rule and the requirement that  $\nabla$  commutes with any contraction, we can extend the action of  $\nabla$  to arbitrary section on  $\mathcal{TE} \setminus \{0\}$ . For a section on  $\mathcal{TE} \setminus \{0\}$  the dynamical covariant derivative is given by  $(\nabla \varphi)(\rho) = \mathcal{S}(\varphi)(\rho) - \varphi(\nabla \rho)$ . For a  $(1, 1)$ -type tensor field  $T$  on  $\mathcal{TE} \setminus \{0\}$  the dynamical covariant derivative has the form

$$(17) \quad \nabla T = \nabla \circ T - T \circ \nabla.$$

and by direct computation using (17) we obtain

$$\nabla h = \nabla v = 0.$$

Also, we get

$$\nabla \mathcal{V}_\beta = v[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{TE}} = - \left( \mathcal{N}_\beta^\alpha + \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} \right) \mathcal{V}_\alpha, \quad \nabla \delta \mathcal{V}^\beta = \left( \mathcal{N}_\alpha^\beta + \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} \right) \delta \mathcal{V}^\alpha,$$

$$\nabla \delta_\beta = h[\mathcal{S}, \delta_\beta]_{\mathcal{TE}} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha, \quad \nabla \mathcal{X}^\beta = - (\mathcal{N}_\alpha^\beta - L_{\alpha\varepsilon}^\beta y^\varepsilon) \mathcal{X}^\alpha.$$

The action of the dynamical covariant derivative on the horizontal section  $X = hX$  is given by following relations

$$(18) \quad \nabla X = \nabla (X^\alpha \delta_\alpha) = \nabla X^\alpha \delta_\alpha, \quad \nabla X^\alpha = \mathcal{S}(X^\alpha) + (\mathcal{N}_\beta^\alpha + y^\varepsilon L_{\varepsilon\beta}^\alpha) X^\beta.$$

**Proposition 3.** *The following results hold*

$$(19) \quad h \circ \mathcal{L}_\mathcal{S} \circ J = -h, \quad J \circ \mathcal{L}_\mathcal{S} \circ v = -v,$$

$$(20) \quad \nabla J = \mathcal{L}_\mathcal{S} J + \mathcal{N}, \quad \nabla J = - \left( \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} - y^\varepsilon L_{\alpha\varepsilon}^\beta + 2\mathcal{N}_\alpha^\beta \right) \mathcal{V}_\beta \otimes \mathcal{X}^\alpha.$$

**Proof.** From (8) we get

$$J[\mathcal{S}, JX]_{\mathcal{T}E} = -JX \Rightarrow J([\mathcal{S}, JX]_{\mathcal{T}E} + X) = 0 \Rightarrow [\mathcal{S}, JX]_{\mathcal{T}E} + X \in V\mathcal{T}E$$

$$h([\mathcal{S}, JX]_{\mathcal{T}E} + X) = 0 \Rightarrow h[\mathcal{S}, JX]_{\mathcal{T}E} = -hX \Leftrightarrow h \circ \mathcal{L}_S \circ J = -h.$$

Also, in  $J[\mathcal{S}, JX]_{\mathcal{T}E} + JX = 0$  considering  $JX = vZ$  it results  $J[\mathcal{S}, vZ]_{\mathcal{T}E} = -vZ \Leftrightarrow J \circ \mathcal{L}_S \circ v = -v$ . Next

$$\begin{aligned} \nabla \circ J &= h \circ \mathcal{L}_S \circ h \circ J + v \circ \mathcal{L}_S \circ v \circ J = v \circ \mathcal{L}_S \circ J = \\ &= (Id - h) \circ \mathcal{L}_S \circ J = \mathcal{L}_S \circ J - h \circ \mathcal{L}_S \circ J = \mathcal{L}_S \circ J + h. \end{aligned}$$

But, on the other hand

$$J \circ \nabla = J \circ \mathcal{L}_S \circ h = J \circ \mathcal{L}_S \circ (Id - v) = J \circ \mathcal{L}_S - J \circ \mathcal{L}_S \circ v = J \circ \mathcal{L}_S + v.$$

and we obtain

$$\nabla \circ J - J \circ \nabla = \mathcal{L}_S \circ J + h - J \circ \mathcal{L}_S - v \Rightarrow \nabla J = \mathcal{L}_S J + h - v = \mathcal{L}_S J + \mathcal{N}.$$

For the last relation, we have

$$\begin{aligned} \nabla J &= \nabla (\mathcal{X}^\beta \otimes \mathcal{V}_\beta) = \nabla \mathcal{X}^\beta \otimes \mathcal{V}_\beta + \mathcal{X}^\beta \otimes \nabla \mathcal{V}_\beta \\ &= -(\mathcal{N}_\alpha^\beta - L_{\alpha\varepsilon}^\beta y^\varepsilon) \mathcal{X}^\alpha \otimes \mathcal{V}_\beta + \mathcal{X}^\beta \otimes \left( -\mathcal{N}_\beta^\alpha - \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} \right) \mathcal{V}_\alpha \\ &= -\mathcal{N}_\alpha^\beta \mathcal{X}^\alpha \otimes \mathcal{V}_\beta + L_{\alpha\varepsilon}^\beta y^\varepsilon \mathcal{X}^\alpha \otimes \mathcal{V}_\beta - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta \otimes \mathcal{V}_\alpha - \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} \mathcal{X}^\beta \otimes \mathcal{V}_\alpha \\ &= \left( L_{\alpha\varepsilon}^\beta y^\varepsilon - \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} - 2\mathcal{N}_\alpha^\beta \right) \mathcal{X}^\alpha \otimes \mathcal{V}_\beta. \end{aligned}$$

□.

The above proposition leads to the following result:

**Theorem 1.** *For a semispray  $\mathcal{S}$ , an arbitrary nonlinear connection  $\mathcal{N}$  and  $\nabla$  dynamical covariant derivative induced by  $\mathcal{S}$  and  $\mathcal{N}$ , the following conditions are equivalent:*

- i)  $\nabla J = 0$ ,
- ii)  $\mathcal{L}_S J + \mathcal{N} = 0$ ,
- iii)  $\mathcal{N}_\alpha^\beta = \frac{1}{2} \left( -\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta \right)$ .

**Proof.** The proof follows from the relations (20). □

This theorem shows that the compatibility condition  $\nabla J = 0$  of the dynamical covariant derivative with the tangent structure determines the nonlinear connection  $\mathcal{N} = -\mathcal{L}_S J$ . For the particular case of tangent bundle we obtain the results from [5]. In the following we deal with this nonlinear connection induced by semispray.

**3.1. The canonical nonlinear connection induced by a semispray.** A semispray  $\mathcal{S}$ , together with the condition  $\nabla J = 0$ , determines an associated nonlinear connection  $\mathcal{N} = -\mathcal{L}_S J$  with local coefficients

$$\mathcal{N}_\alpha^\beta = \frac{1}{2} \left( -\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta \right).$$

In this case the following equations hold

$$[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = -\delta_\beta + (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \mathcal{V}_\alpha,$$



$$[\mathcal{S}, \delta_\beta]_{TE} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha + \mathcal{R}_\beta^\alpha \mathcal{V}_\alpha,$$

where

$$(21) \quad \mathcal{R}_\beta^\alpha = -\sigma_\beta^i \frac{\partial \mathcal{S}^\alpha}{\partial x^i} - \mathcal{S}(\mathcal{N}_\beta^\alpha) - \mathcal{N}_\gamma^\alpha \mathcal{N}_\beta^\gamma + (L_{\varepsilon\beta}^\gamma \mathcal{N}_\gamma^\alpha + L_{\gamma\varepsilon}^\alpha \mathcal{N}_\beta^\gamma) y^\varepsilon.$$

are the local coefficients of the Jacobi endomorphism.

**Definition 5.** *The almost complex structure is given by the formula*

$$\mathbb{F} = h \circ \mathcal{L}_S h - J.$$

We have to show that  $\mathbb{F}^2 = -Id$ . Indeed, from the relation  $\mathcal{L}_S h = \mathcal{L}_S \circ h - h \circ \mathcal{L}_S$  we obtain  $\mathbb{F} = h \circ \mathcal{L}_S \circ h - h \circ \mathcal{L}_S - J = h \circ \mathcal{L}_S \circ (h - Id) - J = -h \circ \mathcal{L}_S \circ v - J$  and  $\mathbb{F}^2 = (-h \circ \mathcal{L}_S \circ v - J) \circ (-h \circ \mathcal{L}_S \circ v - J) = h \circ \mathcal{L}_S \circ v \circ h \circ \mathcal{L}_S \circ v + h \circ \mathcal{L}_S \circ v \circ J + J \circ h \circ \mathcal{L}_S \circ v + J^2 = h \circ \mathcal{L}_S \circ J + J \circ \mathcal{L}_S \circ v = -h - v = -Id$ .

**Proposition 4.** *The following results hold*

$$\begin{aligned} \mathbb{F} \circ J &= h, & J \circ \mathbb{F} &= v, & v \circ \mathbb{F} &= \mathbb{F} \circ h = -J, \\ h \circ \mathbb{F} &= \mathbb{F} \circ v = \mathbb{F} + J, & \mathcal{N} \circ \mathbb{F} &= \mathbb{F} + 2J, & \Phi &= \mathcal{L}_S h - \mathbb{F} - J. \end{aligned}$$

**Proof.** Using the relations (19) we obtain

$$\begin{aligned} \mathbb{F} \circ J &= (-h \circ \mathcal{L}_S \circ v - J) \circ J = -h \circ \mathcal{L}_S \circ v \circ J - J^2 = -h \circ \mathcal{L}_S \circ J = h, \\ J \circ \mathbb{F} &= -J \circ (h \circ \mathcal{L}_S \circ v + J) = -J \circ h \circ \mathcal{L}_S \circ v - J^2 = -J \circ \mathcal{L}_S \circ v = v, \\ v \circ \mathbb{F} &= v \circ (h \circ \mathcal{L}_S h - J) = -v \circ J = -J, \mathbb{F} \circ h = (-h \circ \mathcal{L}_S \circ v - J) \circ h = -J \circ h = -J, \\ h \circ \mathbb{F} &= h \circ (h \circ \mathcal{L}_S h - J) = h \circ \mathcal{L}_S h = \mathbb{F} + J, \mathbb{F} \circ v = (-h \circ \mathcal{L}_S \circ v - J) \circ v = -h \circ \mathcal{L}_S \circ v = \mathbb{F} + J. \end{aligned}$$

In the same way, the other relations can be proved.  $\square$

In local coordinates we have

$$\mathbb{F} = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta \mathcal{V}^\alpha.$$

For a semispray  $\mathcal{S}$  and the associated nonlinear connection we consider the  $\mathbb{R}$ -linear map  $\nabla_0 : \Gamma(\mathcal{T}E \setminus \{0\}) \rightarrow \Gamma(\mathcal{T}E \setminus \{0\})$  given by

$$\nabla_0 \rho = h[\mathcal{S}, h\rho]_{TE} + v[\mathcal{S}, v\rho]_{TE}, \quad \forall \rho \in \Gamma(\mathcal{T}E \setminus \{0\}).$$

It results that

$$\nabla_0(f\rho) = \mathcal{S}(f)\rho + f\nabla_0\rho, \quad \forall f \in C^\infty(E), \quad \rho \in \Gamma(\mathcal{T}E \setminus \{0\})$$

Any tensor derivation on  $\mathcal{T}E \setminus \{0\}$  is completely determined by its actions on smooth functions and sections on  $\mathcal{T}E \setminus \{0\}$  (see [37] generalized Willmore's theorem). Therefore, there exists a unique tensor derivation  $\nabla$  on  $\mathcal{T}E \setminus \{0\}$  such that

$$\nabla|_{C^\infty(E)} = \mathcal{S}, \quad \nabla|_{\Gamma(\mathcal{T}E \setminus \{0\})} = \nabla_0.$$

We will call the tensor derivation  $\nabla$ , the *dynamical covariant derivative* induced by the semispray  $\mathcal{S}$  (see [4] for the tangent bundle case). In this case the dynamical covariant derivative is characterized by the following formulas

$$\begin{aligned} \nabla \mathcal{V}_\beta &= v[\mathcal{S}, \mathcal{V}_\beta]_{TE} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \mathcal{V}_\alpha = -\frac{1}{2} \left( \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} + L_{\beta\varepsilon}^\alpha y^\varepsilon \right) \mathcal{V}_\alpha, \\ \nabla \delta_\beta &= h[\mathcal{S}, \delta_\beta]_{TE} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha = -\frac{1}{2} \left( \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} + L_{\beta\varepsilon}^\alpha y^\varepsilon \right) \delta_\alpha. \end{aligned}$$

**Proposition 5.** *The dynamical covariant derivative has the following decomposition*

$$(22) \quad \nabla = \mathcal{L}_S + \mathbb{F} + J - \Phi.$$

**Proof.** Using the formula (16) and the expressions of  $\mathbb{F}$  and  $\Phi$  we obtain

$$\begin{aligned}
\nabla &= h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v = \\
&= h \circ (\mathcal{L}_S h + h \circ \mathcal{L}_S) + v \circ (\mathcal{L}_S v + v \circ \mathcal{L}_S) = \\
&= h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v + (h + v) \circ \mathcal{L}_S = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v = \\
&= \mathcal{L}_S + \mathbb{F} + J - \Phi.
\end{aligned}$$

**Proposition 6.** *The dynamical covariant derivative induced by the semispray  $\mathcal{S}$  is compatible with  $J$  and  $\mathbb{F}$ , that is*

$$\nabla J = 0, \quad \nabla \mathbb{F} = 0.$$

**Proof.**  $\nabla J = 0$  follows from (20). Using the formula  $\mathbb{F} = -h \circ \mathcal{L}_S \circ v - J$  and  $\nabla \mathbb{F} = \nabla \circ \mathbb{F} - \mathbb{F} \circ \nabla$  we obtain

$$\begin{aligned}
\nabla \mathbb{F} &= (h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v) \circ (-h \circ \mathcal{L}_S \circ v) - (-h \circ \mathcal{L}_S \circ v) \circ (h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v) = \\
&= -h \circ \mathcal{L}_S \circ h \circ \mathcal{L}_S \circ v + h \circ \mathcal{L}_S \circ v \circ \mathcal{L}_S \circ v = \\
&= h \circ \mathcal{L}_S \circ (v - h) \circ \mathcal{L}_S \circ v = h \circ \mathcal{L}_S \circ \mathcal{L}_S J \circ \mathcal{L}_S \circ v = \\
&= h \circ \mathcal{L}_S \circ (\mathcal{L}_S \circ J - J \circ \mathcal{L}_S) \circ \mathcal{L}_S \circ v = \\
&= h \circ \mathcal{L}_S \circ \mathcal{L}_S \circ (J \circ \mathcal{L}_S \circ v) - (h \circ \mathcal{L}_S \circ J) \circ \mathcal{L}_S \circ \mathcal{L}_S \circ v = \\
&= -h \circ \mathcal{L}_S \circ \mathcal{L}_S \circ v + h \circ \mathcal{L}_S \circ \mathcal{L}_S \circ v = 0.
\end{aligned}$$

□

**Proposition 7.** *If the dynamical covariant derivative is induced by a spray  $\mathcal{S}$  then*

$$\nabla \mathcal{S} = 0, \quad \nabla \mathbb{C} = 0.$$

**Proof.** Indeed, if  $\mathcal{S}$  is a spray we have  $\mathcal{S} = h\mathcal{S}$  and  $v\mathcal{S} = 0$  and it results  $\nabla \mathcal{S} = h \circ \mathcal{L}_S \circ h\mathcal{S} + v \circ \mathcal{L}_S \circ v\mathcal{S} = h \circ \mathcal{L}_S \circ \mathcal{S} = 0$ . Also  $\nabla \mathbb{C} = 0$  follows from  $h\mathbb{C} = 0$ ,  $v\mathbb{C} = \mathbb{C}$  and  $[\mathbb{C}, \mathcal{S}]_{TE} = \mathcal{S}$ . □

Other geometric structure induced by a nonlinear connection is the Berwald connection

$$\mathcal{D} : \Gamma(\mathcal{TE} \setminus \{0\}) \times \Gamma(\mathcal{TE} \setminus \{0\}) \rightarrow \Gamma(\mathcal{TE} \setminus \{0\})$$

$$\mathcal{D}_X Y = v[hX, vY]_{TE} + h[vX, hY]_{TE} + J[vX, (\mathbb{F} + J)Y]_{TE} + (\mathbb{F} + J)[hX, JY]_{TE}.$$

**Proposition 8.** *The Berwald connection has the following properties*

$$\mathcal{D}h = 0, \quad \mathcal{D}v = 0, \quad \mathcal{D}J = 0, \quad \mathcal{D}\mathbb{F} = 0$$

**Proof.** Using the properties of the vertical and horizontal projectors we obtain

$$\mathcal{D}_X vY = v[hX, vY]_{TE} + J[vX, (\mathbb{F} + J)Y]_{TE} \text{ and}$$

$$v(\mathcal{D}_X Y) = v[hX, vY]_{TE} + J[vX, (\mathbb{F} + J)Y]_{TE} \text{ which yields } \mathcal{D}v = 0. \text{ Also,}$$

$$\mathcal{D}_X hY = h[vX, hY]_{TE} + (\mathbb{F} + J)[hX, JY]_{TE} = h(\mathcal{D}_X Y) \text{ and it results } \mathcal{D}h = 0.$$

Moreover,

$$\mathcal{D}_X JY = v[hX, JY]_{TE} + J[vX, hY]_{TE} \text{ and } J(\mathcal{D}_X Y) = J[vX, hY]_{TE} + v[hX, JY]_{TE}$$

and we obtain  $\mathcal{D}J = 0$ . From

$$\mathcal{D}_X \mathbb{F}Y = v[hX, -JY]_{TE} + h[vX, (\mathbb{F} + J)Y]_{TE} + J[vX, -hY]_{TE} + (\mathbb{F} + J)[hX, vY]_{TE}$$

and

$$\mathbb{F}(\mathcal{D}_X Y) = (\mathbb{F} + J)[hX, vY]_{TE} - J[vX, hY]_{TE} + h[vX, (\mathbb{F} + J)Y]_{TE} - v[hX, JY]_{TE} = \mathcal{D}_X \mathbb{F}Y \text{ we get } \mathcal{D}\mathbb{F} = 0.$$

□

In local coordinates we obtain

$$\mathcal{D}_{\delta_\alpha} \delta_\beta = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} \delta_\gamma, \quad \mathcal{D}_{\delta_\alpha} \mathcal{V}_\beta = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} \mathcal{V}_\gamma, \quad \mathcal{D}_{\mathcal{V}_\alpha} \delta_\beta = 0, \quad \mathcal{D}_{\mathcal{V}_\alpha} \mathcal{V}_\beta = 0.$$

**Proposition 9.** *If  $\mathcal{S}$  is a spray then the following equality hold*

$$\nabla = \mathcal{D}_\mathcal{S}.$$

**Proof.** If  $\mathcal{S}$  is a spray then  $\mathcal{S} = \mathbf{h}\mathcal{S}$  and  $\mathbf{v}\mathcal{S} = 0$  which implies

$$\mathcal{D}_\mathcal{S} Y = \mathbf{v}[\mathcal{S}, \mathbf{v}Y]_{\mathcal{T}E} + (\mathbb{F} + J)[\mathcal{S}, JY]_{\mathcal{T}E}.$$

But  $\nabla Y = \mathbf{h}[\mathcal{S}, \mathbf{h}Y]_{\mathcal{T}E} + \mathbf{v}[\mathcal{S}, \mathbf{v}Y]_{\mathcal{T}E}$  and we will prove that  $\mathbf{h}[\mathcal{S}, \mathbf{h}Y]_{\mathcal{T}E} = (\mathbb{F} + J)[\mathcal{S}, JY]_{\mathcal{T}E}$  using the computation in local coordinates. Let us consider  $Y = X^\alpha(x, y)\mathcal{X}_\alpha + Y^\beta(x, y)\mathcal{V}_\beta$  and using (10) we get

$$[\mathcal{S}, \mathbf{h}Y]_{\mathcal{T}E} = [y^\alpha \delta_\alpha, X^\beta \delta_\beta]_{\mathcal{T}E} = y^\alpha X^\beta \mathcal{R}_{\alpha\beta}^\varepsilon \mathcal{V}_\varepsilon + y^\alpha X^\beta L_{\alpha\beta}^\varepsilon \delta_\varepsilon + y^\alpha \delta_\alpha (X^\beta) \delta_\beta + X^\beta N_\beta^\alpha \delta_\alpha$$

But  $L_{\alpha\beta}^\varepsilon = -L_{\beta\alpha}^\varepsilon$  and it results  $y^\alpha X^\beta L_{\alpha\beta}^\varepsilon = 0$  which leads to

$$\mathbf{h}[\mathcal{S}, \mathbf{h}Y]_{\mathcal{T}E} = (y^\alpha \delta_\alpha (X^\beta) + X^\alpha N_\alpha^\beta) \delta_\beta.$$

Next

$$[\mathcal{S}, JY]_{\mathcal{T}E} = [y^\alpha \delta_\alpha, X^\beta \mathcal{V}_\beta]_{\mathcal{T}E} = y^\alpha X^\beta \frac{\partial N_\alpha^\varepsilon}{\partial y^\beta} \mathcal{V}_\varepsilon + y^\alpha \delta_\alpha (X^\beta) \mathcal{V}_\beta - X^\beta \delta_\beta.$$

From (15) we have

$$y^\alpha X^\beta \frac{\partial N_\alpha^\varepsilon}{\partial y^\beta} = \mathcal{N}_\beta^\varepsilon X^\beta - L_{\beta\alpha}^\varepsilon y^\alpha X^\beta = \mathcal{N}_\beta^\varepsilon X^\beta,$$

and using the relations  $(\mathbb{F} + J)(\mathcal{V}_\alpha) = \delta_\alpha$ ,  $(\mathbb{F} + J)(\delta_\alpha) = 0$  we obtain the result which ends the proof.  $\square$

#### 4. Symmetries for semispray

In this section we study the symmetries of SODE on Lie algebroids and prove that the nonlinear connection can be determined by these symmetries. In the particular case of tangent bundle some results from [5] are obtained.

**Definition 6.** *A section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is a dynamical symmetry of semispray  $\mathcal{S}$  if  $[\mathcal{S}, X]_{\mathcal{T}E} = 0$ .*

In local coordinates for  $X = X^\alpha(x, y)\mathcal{X}_\alpha + Y^\alpha(x, y)\mathcal{V}_\alpha$  we obtain

$$[\mathcal{S}, X]_{\mathcal{T}E} = (y^\alpha L_{\alpha\gamma}^\beta X^\gamma - Y^\beta + \mathcal{S}(X^\beta)) \mathcal{X}_\beta + (\mathcal{S}(Y^\beta) - X(\mathcal{S}^\beta)) \mathcal{V}_\beta,$$

and it results that the dynamical symmetry is characterized by the equations

$$(23) \quad Y^\alpha = \mathcal{S}(X^\alpha) + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta,$$

$$(24) \quad \mathcal{S}(Y^\alpha) - X(\mathcal{S}^\alpha) = 0.$$

Introducing (23) into (24) we obtain

$$\mathcal{S}^2(X^\alpha) - X(\mathcal{S}^\alpha) = \left( \sigma_\gamma^i \frac{\partial L_{\varepsilon\beta}^\alpha}{\partial x^i} X^\beta + L_{\varepsilon\beta}^\alpha \sigma_\gamma^i \frac{\partial X^\beta}{\partial x^i} \right) y^\gamma y^\varepsilon + \mathcal{S}^\gamma \left( L_{\gamma\beta}^\alpha X^\beta + y^\varepsilon L_{\varepsilon\beta}^\alpha \frac{\partial X^\beta}{\partial y^\gamma} \right).$$

**Definition 7.** *A section  $X = \tilde{X}^\alpha(x, y)s_\alpha$  on  $E \setminus \{0\}$  is a Lie symmetry of a semispray if its complete lift  $\tilde{X}^c$  is a dynamical symmetry, that is  $[\mathcal{S}, \tilde{X}^c]_{\mathcal{T}E} = 0$ .*

**Proposition 10.** *The local expression of a Lie symmetry is given by*

$$\mathcal{S}^\alpha \frac{\partial \tilde{X}^\beta}{\partial y^\alpha} = 0,$$

$$\mathcal{S}^\alpha \tilde{X}_{|\alpha}^\beta + y^\alpha y^\varepsilon \sigma_\alpha^i \frac{\partial \tilde{X}_{|\varepsilon}^\beta}{\partial x^i} - \tilde{X}^\alpha \sigma_\alpha^i \frac{\partial \mathcal{S}^\beta}{\partial x^i} - y^\varepsilon \tilde{X}_{|\varepsilon}^\alpha \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + \mathcal{S}^\alpha y^\varepsilon \left( \sigma_\varepsilon^i \frac{\partial^2 \tilde{X}^\beta}{\partial y^\alpha \partial x^i} - L_{\gamma\varepsilon}^\beta \frac{\partial \tilde{X}^\gamma}{\partial y^\alpha} \right) = 0.$$

where

$$\tilde{X}_{|\varepsilon}^\alpha := \sigma_\varepsilon^i \frac{\partial \tilde{X}^\alpha}{\partial x^i} - L_{\beta\varepsilon}^\alpha \tilde{X}^\beta,$$

**Proof.** Considering  $\tilde{X}^c = \tilde{X}^\alpha \mathcal{X}_\alpha + y^\varepsilon \tilde{X}_{|\varepsilon}^\alpha \mathcal{V}_\alpha$  and using (1) we obtain

$$\begin{aligned} [\mathcal{S}, \tilde{X}^c]_{\mathcal{T}E} &= \left( \tilde{X}^\alpha y^\varepsilon L_{\varepsilon\alpha}^\beta + y^\alpha \sigma_\alpha^i \frac{\partial \tilde{X}^\beta}{\partial x^i} - y^\varepsilon \tilde{X}_{|\varepsilon}^\beta + \mathcal{S}^\alpha \frac{\partial \tilde{X}^\beta}{\partial y^\alpha} \right) \mathcal{X}_\beta + \\ &\quad \left( y^\alpha y^\varepsilon \sigma_\alpha^i \frac{\partial \tilde{X}_{|\varepsilon}^\beta}{\partial x^i} - \tilde{X}^\alpha \sigma_\alpha^i \frac{\partial \mathcal{S}^\beta}{\partial x^i} + \mathcal{S}^\alpha \tilde{X}_{|\alpha}^\beta - y^\varepsilon \tilde{X}_{|\varepsilon}^\alpha \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + \mathcal{S}^\alpha y^\varepsilon \left( \sigma_\varepsilon^i \frac{\partial^2 \tilde{X}^\beta}{\partial y^\alpha \partial x^i} - L_{\gamma\varepsilon}^\beta \frac{\partial \tilde{X}^\gamma}{\partial y^\alpha} \right) \right) \mathcal{V}_\beta. \end{aligned}$$

We deduce that  $\tilde{X}^\alpha y^\varepsilon L_{\varepsilon\alpha}^\beta + y^\alpha \sigma_\alpha^i \frac{\partial \tilde{X}^\beta}{\partial x^i} - y^\varepsilon \tilde{X}_{|\varepsilon}^\beta = 0$  and it results the local expression of a Lie symmetry.  $\square$

We have to remark that a section  $\tilde{X} = \tilde{X}^\alpha(x) s_\alpha$  on  $E \setminus \{0\}$  is a Lie symmetry if and only if (see also [29])

$$y^\alpha y^\varepsilon \sigma_\alpha^i \frac{\partial \tilde{X}_{|\varepsilon}^\beta}{\partial x^i} - \tilde{X}^\alpha \sigma_\alpha^i \frac{\partial \mathcal{S}^\beta}{\partial x^i} + \mathcal{S}^\alpha \tilde{X}_{|\alpha}^\beta - y^\varepsilon \tilde{X}_{|\varepsilon}^\alpha \frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} = 0.$$

and it results, by direct computation, that the components  $\tilde{X}^\alpha(x)$  satisfy the equations (23), (24).

**Definition 8.** *A section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is called newtonoid if  $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$ .*

In local coordinates we obtain

$$J[\mathcal{S}, X]_{\mathcal{T}E} = (\mathcal{S}(X^\alpha) - Y^\alpha + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta) \mathcal{X}_\alpha,$$

which yields

$$(25) \quad Y^\alpha = \mathcal{S}(X^\alpha) + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta, \quad X = X^\alpha \mathcal{X}_\alpha + (\mathcal{S}(X^\alpha) + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta) \mathcal{V}_\alpha.$$

We have to remark that a section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is a dynamical symmetry if and only if it is a newtonoid and satisfies the equation (24). The set of newtonoid sections denoted  $\mathfrak{X}_{\mathcal{S}}$  is given by

$$\mathfrak{X}_{\mathcal{S}} = \text{Ker}(J \circ \mathcal{L}_{\mathcal{S}}) = \text{Im}(Id + J \circ \mathcal{L}_{\mathcal{S}}).$$

In the following we will use the dynamical covariant derivative in order to characterize the newtonoid section on Lie algebroids. Let  $\mathcal{S}$  a semispray,  $\mathcal{N}$  an arbitrary nonlinear connection and  $\nabla$  the induced dynamical covariant derivative. We set:

**Proposition 11.** *A section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is a newtonoid if and only if*

$$(26) \quad v(X) = J(\nabla X),$$

which locally yields

$$X = X^\alpha \delta_\alpha + \nabla X^\alpha \mathcal{V}_\alpha$$

with  $\nabla X^\alpha$  given by formula (18).

**Proof.** We know that  $J \circ \nabla = J \circ \mathcal{L}_S + v$  and it results  $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$  if and only if  $v(X) = J(\nabla X)$ . In local coordinates we obtain

$$\begin{aligned} X &= X^\alpha (\delta_\alpha + \mathcal{N}_\alpha^\beta \mathcal{V}_\beta) + (\mathcal{S}(X^\alpha) + y^\varepsilon L_{\varepsilon\beta}^\alpha X^\beta) \mathcal{V}_\alpha \\ &= X^\alpha \delta_\alpha + (\mathcal{S}(X^\alpha) + X^\beta (\mathcal{N}_\beta^\alpha + y^\varepsilon L_{\varepsilon\beta}^\alpha)) \mathcal{V}_\alpha \\ &= X^\alpha \delta_\alpha + \nabla X^\alpha \mathcal{V}_\alpha. \end{aligned}$$

□

**Proposition 12.** *A section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is a dynamical symmetry if and only if  $X$  is a newtonoid and*

$$(27) \quad \nabla(J\nabla X) + \Phi(X) = 0.$$

**Proof.** If  $X$  is a dynamical symmetry then  $h[\mathcal{S}, X]_{\mathcal{T}E} = v[\mathcal{S}, X]_{\mathcal{T}E} = 0$  and composing by  $J$  we get  $J[\mathcal{S}, X]_{\mathcal{T}E} = 0$  that means  $X$  is a newtonoid. Therefore,  $v[\mathcal{S}, X]_{\mathcal{T}E} = v[\mathcal{S}, vX]_{\mathcal{T}E} + v[\mathcal{S}, hX]_{\mathcal{T}E} = \nabla(vX) + \Phi(X)$  and using (26) we get  $\nabla(J\nabla X) + \Phi(X) = 0$ . □

**Definition 9.** *A section  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  is called a Cartan symmetry of the Lagrangian  $L$ , if  $\mathcal{L}_X \omega_L = 0$  and  $\mathcal{L}_X E_L = 0$ .*

**Proposition 13.** *Consider  $L$  a regular Lagrangian on  $\mathcal{T}E$  and  $X \in \Gamma(\mathcal{T}E \setminus \{0\})$  a Cartan symmetry of  $L$ . Then  $X$  is a Newtonoid section for the semispray induced by  $L$ .*

**Proof.** From the symplectic equation  $i_S \omega_L = -d^E E_L$ , applying the Lie derivative in the both sides, we obtain

$$\mathcal{L}_X i_S \omega_L = -\mathcal{L}_X d^E E_L = -d^E \mathcal{L}_X E_L = 0.$$

Also, using the formula  $i_{[X, Y]}_{\mathcal{T}E} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X$  it results

$$\mathcal{L}_X i_S \omega_L = -i_{[S, X]}_{\mathcal{T}E} \omega_L + i_S \mathcal{L}_X \omega_L = -i_{[S, X]}_{\mathcal{T}E} \omega_L$$

which yields

$$(28) \quad i_{[S, X]}_{\mathcal{T}E} \omega_L = 0.$$

For  $X = X^\alpha(x, y) \mathcal{X}_\alpha + Y^\alpha(x, y) \mathcal{V}_\alpha$  we have

$$(29) \quad [\mathcal{S}, X]_{\mathcal{T}E} = V^\beta \mathcal{X}_\beta + W^\beta \mathcal{V}_\beta,$$

where  $V^\beta = y^\alpha L_{\alpha\gamma}^\beta X^\gamma - Y^\beta + \mathcal{S}(X^\beta)$  and  $W^\beta = \mathcal{S}(Y^\beta) - X(\mathcal{S}^\beta)$ . Also the symplectic structure induced by the regular Lagrangian  $L$  can be written

$$(30) \quad \omega_L = a_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta + g_{\alpha\beta} \mathcal{V}^\beta \wedge \mathcal{X}^\alpha,$$

with

$$a_{\alpha\beta} = \frac{1}{2} \left( \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right).$$

Replacing (29) and (30) into (28) by straightforward computation we obtain

$$(2a_{\alpha\beta} V^\beta + g_{\alpha\beta} W^\beta) \mathcal{X}^\alpha + g_{\alpha\beta} V^\beta \mathcal{V}^\alpha = 0,$$

and it results  $g_{\alpha\beta} V^\beta = 0$ . But  $\text{rank} g_{\alpha\beta} = m$  which implies that  $V^\beta = 0$  and  $Y^\beta = y^\alpha L_{\alpha\gamma}^\beta X^\gamma + \mathcal{S}(X^\beta)$  and we obtain that  $X$  is a Newtonoid section for  $\mathcal{S}$ . □

For  $f \in C^\infty(\mathcal{TE})$  and  $X \in \Gamma(\mathcal{TE} \setminus \{0\})$  we define the product

$$f * X = (Id + J \circ \mathcal{L}_S)(fX) = fX + fJ[\mathcal{S}, X]_{\mathcal{TE}} + \mathcal{S}(f)JX,$$

and remark that a section  $X \in \Gamma(\mathcal{TE} \setminus \{0\})$  is a newtonoid if and only if

$$X = X^\alpha(x, y) * \mathcal{X}_\alpha.$$

If  $X \in \mathfrak{X}_S$  then

$$f * X = fX + \mathcal{S}(f)JX.$$

**Proposition 14.** *Let us consider a semispray  $\mathcal{S}$ , an arbitrary nonlinear connection  $\mathcal{N}$  and  $\nabla$  the dynamical covariant derivative. The following conditions are equivalent:*

i)  $\nabla$  restricts to  $\nabla : \mathfrak{X}_S \rightarrow \mathfrak{X}_S$  satisfies the Leibniz rule with respect to the  $*$  product.

ii)  $\nabla J = 0$ ,

iii)  $\mathcal{L}_S J + \mathcal{N} = 0$ ,

iv)  $\mathcal{N}_\alpha^\beta = \frac{1}{2} \left( -\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta \right)$ .

**Proof.** For  $ii) \Rightarrow i)$  we consider  $X \in \mathfrak{X}_S$  and using (26) we have  $vX = J(\nabla X)$  which leads to  $\nabla(vX) = \nabla(J\nabla X)$ . It results  $(\nabla v)X + v(\nabla X) = (\nabla J)(\nabla X) + J\nabla(\nabla X)$  and using the relations  $\nabla v = 0$  and  $\nabla J = 0$  we obtain  $v(\nabla X) = J\nabla(\nabla X)$  which implies  $\nabla X \in \mathfrak{X}_S$ . For  $X \in \mathfrak{X}_S$  we have

$$\nabla(f * X) = \nabla(fX + \mathcal{S}(f)JX) = \mathcal{S}(f)X + f\nabla X + \mathcal{S}^2(f)JX + \mathcal{S}(f)\nabla(JX),$$

$$\nabla f * X + f * \nabla X = \mathcal{S}(f)X + \mathcal{S}^2(f)JX + f\nabla X + \mathcal{S}(f)J(\nabla X).$$

But  $\nabla(JX) = (\nabla J)X + J(\nabla X)$  and from  $\nabla J = 0$  we obtain  $\nabla(JX) = J(\nabla X)$  which leads to  $\nabla(f * X) = \nabla f * X + f * \nabla X$ .

For  $i) \Rightarrow ii)$  we consider the set  $\mathfrak{X}_S \cup \Gamma^v(\mathcal{TE} \setminus \{0\})$  which is a set of generators for  $\Gamma(\mathcal{TE} \setminus \{0\})$ . We have  $\nabla J(X) = 0$  for  $X \in \Gamma^v(\mathcal{TE} \setminus \{0\})$  and for  $X \in \mathfrak{X}_S$  using  $\nabla(f * X) = \nabla f * X + f * \nabla X$  it results  $\mathcal{S}(f)\nabla(JX) = \mathcal{S}(f)J(\nabla X)$ , which implies  $\mathcal{S}(f)(\nabla J)X = 0$ , for an arbitrary function  $f \in C^\infty(\mathcal{TE} \setminus \{0\})$ . Therefore,  $\nabla J = 0$  on  $\mathfrak{X}_S$  which ends the proof. The equivalence of the conditions  $ii)$ ,  $iii)$ ,  $iv)$  have been proved in the theorem 3.2  $\square$

Next we consider the dynamical covariant derivative  $\nabla$  induced by the semispray  $\mathcal{S}$  and associated nonlinear connection  $\mathcal{N} = -\mathcal{L}_S J$ .

**Proposition 15.** *A section  $X \in \Gamma(\mathcal{TE} \setminus \{0\})$  is a dynamical symmetry if and only if  $X$  is a newtonoid and*

$$(31) \quad \nabla^2 JX + \Phi(X) = 0,$$

which locally yields

$$\nabla^2 X^\alpha + \mathcal{R}_\beta^\alpha X^\beta = 0.$$

**Proof.** From (20) it results  $\nabla J = 0$  and using (27) and (17) we get (31). Next, using (25) and (14) the local components of the vertical section  $\nabla^2 JX + \Phi(X)$  is  $\nabla^2 X^\alpha + \mathcal{R}_\beta^\alpha X^\beta$ .  $\square$

**Proposition 16.** *A section  $\tilde{X} \in \Gamma(E \setminus \{0\})$  is a Lie symmetry of  $\mathcal{S}$  if and only if*

$$(32) \quad \nabla^2 \tilde{X}^v + \Phi(\tilde{X}^c) = 0$$

**Proof.** Using (31) and the relation  $J(\tilde{X}^c) = \tilde{X}^v$  we obtain (32).  $\square$

**Conclusions.** The main purpose of this paper is to study the symmetries of SODE on Lie algebroids and relations between them, using the dynamical covariant derivative and Jacobi endomorphism. The existence of a semispray  $\mathcal{S}$  together with an arbitrary nonlinear connection  $\mathcal{N}$  define a dynamical covariant derivative and the Jacobi endomorphism. In the case of homogeneous SODE (spray), the Jacobi endomorphism is the contraction with  $\mathcal{S}$  of the curvature of the nonlinear connection. Let us remark that at this point we do not have any relation between  $\mathcal{S}$  and the nonlinear connection  $\mathcal{N}$ . This will be given considering the compatibility condition between the dynamical covariant derivative and the tangent structure,  $\nabla J = 0$ , which fix the canonical nonlinear connection  $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ . This canonical nonlinear connection depends only on semispray. In this case we have the decomposition  $\nabla = \mathcal{L}_{\mathcal{S}} + \mathbb{F} + \mathcal{J} - \Phi$  which can be compared with the tangent case from [5, 25]. Also, in the case of spray, the dynamical covariant derivative coincides with Berwald connection. We study the dynamical symmetry, Lie symmetry, Newtonoid section and Cartan symmetry on Lie algebroids and characterize their properties with the help of dynamical covariant derivative and Jacobi endomorphism. As further developments one can study the symmetries using the  $k$ -symplectic formalism on Lie algebroids given in [21].

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